

Diagonal approximation and the cohomology ring of the fundamental groups of surfaces

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Abstract

We construct finite free resolutions of \mathbb{Z} over $\mathbb{Z}\pi$, where π is the fundamental group of a surface distinct from S^2 and $\mathbb{R}P^2$, and define diagonal approximations for these resolutions. We then proceed to give some possible applications that come with the knowledge of those maps.

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1 Introduction

The closed surfaces other than S^2 and $\mathbb{R}P^2$ are $K(\pi, 1)$ spaces, which means that their cohomology rings coincide with the cohomology rings of their fundamental groups. There are many situations where one may be interested in the cohomology groups of the surfaces. See, for instance, [5]. Given that interest, in this paper we define finite free resolutions for the fundamental groups of the $K(\pi, 1)$ surfaces and partial diagonal approximations for these resolutions, which allow us to compute the cohomology rings $H^*(\pi, M)$ for any coefficient M in an efficient way.

Let us briefly recall how to define and compute (at least in theory) not only the cohomology groups $H^*(\pi, M)$ for a given group π with coefficients in the $\mathbb{Z}\pi$ -module M , but also how to determine the multiplicative structure given by the cup product. More details about the definitions can be found in [2].

If M is a (left) $\mathbb{Z}\pi$ -module, the cohomology group $H^n(\pi, M)$ is defined by $H^n(\pi, M) = \text{Ext}_{\mathbb{Z}\pi}^n(\mathbb{Z}, M)$ for $n \geq 0$, where \mathbb{Z} is the trivial $\mathbb{Z}\pi$ -module. Hence one way to compute the groups $H^n(\pi, M)$ is this: first, we find a projective resolution P over the ring $\mathbb{Z}\pi$ of the trivial $\mathbb{Z}\pi$ -module \mathbb{Z} . Then we apply the functor $\text{Hom}_{\mathbb{Z}\pi}(_, M)$ to the chain complex P and the cohomology groups $H^n(\pi, M)$ are the cohomology groups of the chain complex $\text{Hom}_{\mathbb{Z}\pi}(P, M)$. Also, if

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a projective resolution of \mathbb{Z} , then

$$\cdots \longrightarrow (P \otimes P)_n \xrightarrow{\partial_n} (P \otimes P)_{n-1} \longrightarrow \cdots \longrightarrow (P \otimes P)_0 \xrightarrow{\varepsilon \otimes \varepsilon} \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \longrightarrow 0$$

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is also a projective resolution, and there is a map of chain complexes $\Delta: P \rightarrow (P \otimes P)$ such that

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_n & \xrightarrow{d_n} & P_{n-1} & \longrightarrow & \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \\ & & \downarrow \Delta_n & & \downarrow \Delta_{n-1} & & \downarrow \Delta_0 \\ \cdots & \longrightarrow & (P \otimes P)_n & \xrightarrow{\partial_n} & (P \otimes P)_{n-1} & \longrightarrow & \cdots \longrightarrow (P \otimes P)_0 \xrightarrow{\varepsilon \otimes \varepsilon} \mathbb{Z} \longrightarrow 0 \end{array}$$

is commutative. The map Δ is called a *diagonal approximation* for the resolution P and is used to define the cup product

$$H^p(\pi, M) \otimes H^q(\pi, N) \xrightarrow{\smile} H^{p+q}(\pi, M \otimes N)$$

in the following way: let $u \in \text{Hom}_{\mathbb{Z}\pi}(P_p, M)$, $v \in \text{Hom}_{\mathbb{Z}\pi}(P_q, N)$ and let $\alpha \in H^p(G, M)$ and $\beta \in H^q(\pi, N)$ be the classes of the homomorphisms u and v , respectively. The *cup product* of α and β is defined by

$$(\alpha \smile \beta) = [(u \otimes v) \circ \Delta] \in H^{p+q}(\pi, M \otimes N), \quad (1)$$

where $[\]$ denotes the cohomology class. Given that definition, we can be a little more specific: the product $(\alpha \smile \beta)$ in (1) is the cohomology class of the map $(u \otimes v) \circ \Delta_{pq}$, where $\Delta_{pq}: P_{p+q} \rightarrow P_p \otimes P_q$ is the composition of $\Delta_{p+q}: P_{p+q} \rightarrow (P \otimes P)_{p+q}$ with the projection $\pi_{pq}: (P \otimes P)_{p+q} \rightarrow (P_p \otimes P_q)$.

Thus, the computation of the cohomology groups $H^*(\pi, M)$ together with the multiplicative structure given by the cup product can be accomplished if we manage to find the free resolution P and the diagonal approximation Δ for the resolution P . This is no easy task in general. In [10], we find the following two propositions, which can help us determine Δ and were used by the authors to compute the cohomology of certain 4-periodic groups:

Proposition 1 *For a group π , let*

$$\cdots \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

be a finitely generated free resolution of \mathbb{Z} over $\mathbb{Z}\pi$, that is, each C_n is finitely generated as a $\mathbb{Z}\pi$ -module. If s is a contracting homotopy for the resolution C , then a contracting homotopy \tilde{s} for the free resolution $C \otimes C$ of \mathbb{Z} over $\mathbb{Z}\pi$ is given by

$$\begin{aligned} \tilde{s}_{-1}: \mathbb{Z} &\rightarrow C_0 \otimes C_0 \\ \tilde{s}_{-1}(1) &= s_{-1}(1) \otimes s_{-1}(1), \\ \tilde{s}_n: (C \otimes C)_n &\rightarrow (C \otimes C)_{n+1} \\ \tilde{s}_n(u_i \otimes v_{n-i}) &= s_i(u_i) \otimes v_{n-i} + s_{-1}\varepsilon(u_i) \otimes s_{n-i}(v_{n-i}), \quad \text{if } n \geq 0, \end{aligned}$$

where $s_{-1}\varepsilon: C_0 \rightarrow C_0$ is extended to $s_{-1}\varepsilon = \{(s_{-1}\varepsilon)_n: C_n \rightarrow C_n\}$ in such a way that $(s_{-1}\varepsilon)_n = 0$ for $n \geq 1$.

Proposition 2 *For a group π , let*

$$\cdots \longrightarrow C_n \xrightarrow{d_n} \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

be a finitely generated free resolution of \mathbb{Z} over $\mathbb{Z}\pi$ (i.e., each C_n is a finitely generated free $\mathbb{Z}\pi$ -module), and let s be a contracting homotopy for this resolution C . If \tilde{s} is the contracting homotopy for the

resolution $C \otimes C$ given by Proposition 1, then a diagonal approximation $\Delta: C \rightarrow C \otimes C$ can be defined in the following way: for each $n \geq 0$, the map $\Delta_n: C_n \rightarrow (C \otimes C)_n$ is given in each generator ρ of C_n by

$$\begin{aligned}\Delta_0 &= s_{-1}\varepsilon \otimes s_{-1}\varepsilon, \\ \Delta_n(\rho) &= \tilde{s}_{n-1}\Delta_{n-1}d_n(\rho), \quad \text{if } n \geq 1.\end{aligned}$$

Essentially, the above propositions tell us that if we have a finitely generated free resolution P of \mathbb{Z} over $\mathbb{Z}\pi$, then a diagonal approximation $\Delta: P \rightarrow (P \otimes P)$ can be calculated if we have a contracting homotopy for the resolution P . This is the approach we will use to determine partial diagonal approximations for the resolutions of the fundamental groups of the $K(\pi, 1)$ surfaces. The word “partial” has the following meaning: since we are dealing with $K(\pi, 1)$ surfaces, the finitely generated free resolution P is given by the augmented cellular chain complex of the universal cover \mathbb{R}^2 of the surface as a free π -complex. As the comohology groups $H^n(\pi, M)$ are then trivial for $n \geq 3$, the observation after (1) tells that we only need the maps $\Delta_0: P_0 \rightarrow (P_0 \otimes P_0)$, $\Delta_{01}: P_1 \rightarrow (P_0 \otimes P_1)$, $\Delta_{02}: P_2 \rightarrow (P_0 \otimes P_2)$ and $\Delta_{11}: P_2 \rightarrow (P_1 \otimes P_1)$ to compute the meaningful products $H^p(\pi, M) \otimes H^q(\pi, N) \xrightarrow{\sim} H^{p+q}(\pi, M \otimes N)$.

In Section 2 we'll use the results above to determine diagonal approximations for the fundamental groups of the $K(\pi, 1)$ surfaces, and in Section 3 we make some comments on applications.

2 Free resolutions and diagonal approximations

Let M_n be the orientable surface of genus $n \geq 1$, with fundamental group G given by the presentation

$$G = \pi_1(M_n) = \langle a_1, b_1, a_2, b_2, \dots, a_n, b_n \mid [a_1, b_1][a_2, b_2] \cdots [a_n, b_n] \rangle.$$

We also let $p_i = [a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ and $p = p_1 p_2 \cdots p_n$.

Proposition 3 (Free resolution for the orientable case) *A free resolution of \mathbb{Z} over $\mathbb{Z}G$ is given by*

$$0 \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \quad (2)$$

where

$$\begin{aligned}P_0 &= \langle x \rangle \cong \mathbb{Z}G, \\ P_1 &= \langle y_1, \dots, y_n, z_1, \dots, z_n \rangle \cong \mathbb{Z}G^{2n}, \\ P_2 &= \langle w \rangle \cong \mathbb{Z}G,\end{aligned}$$

and the maps ε , d_1 and d_2 are defined by

$$\begin{aligned}\varepsilon(x) &= 1, \\ d_1(y_i) &= (a_i - 1)x, \\ d_1(z_i) &= (b_i - 1)x, \\ d_2(w) &= \sum_{i=1}^n \left(\frac{\partial p}{\partial a_i} y_i + \frac{\partial p}{\partial b_i} z_i \right),\end{aligned}$$

where the partial derivatives are the Fox derivatives.

Proof: Since the surface M_n is a $K(G, 1)$ space, its universal cover \mathbb{R}^2 is a free and contractible G -complex. Hence its augmented cellular chain complex, which is proved to be exactly 2 in [3], is a free resolution of \mathbb{Z} over $\mathbb{Z}G$. ■

We now proceed to construct a diagonal approximation $\Delta: P \rightarrow (P \otimes P)$ for the resolution P given by the above proposition.

Theorem 1 (Diagonal approximation for the orientable case) *Let P be the free resolution of \mathbb{Z} over $\mathbb{Z}G$ given by Proposition 3. A diagonal approximation $\Delta: P \rightarrow (P \otimes P)$ is partially given by*

$$\begin{aligned}
\Delta_0: P_0 &\rightarrow (P \otimes P)_0 \\
\Delta_0(x) &= x \otimes x, \\
\Delta_1: P_1 &\rightarrow (P \otimes P)_1 \\
\Delta_1(y_i) &= y_i \otimes a_i x + x \otimes y_i, \\
\Delta_1(z_i) &= z_i \otimes b_i x + x \otimes z_i, \\
\Delta_{02}: P_2 &\rightarrow (P_0 \otimes P_2) \\
\Delta_{02}(w) &= x \otimes w, \\
\Delta_{11}: P_2 &\rightarrow (P_1 \otimes P_1) \\
\Delta_{11}(w) &= \sum_{i=1}^n \left[\sum_{j=1}^{i-1} (p_1 \cdots p_{j-1}) (1 - a_j b_j a_j^{-1}) y_j \otimes (p_1 \cdots p_{i-1}) y_i + \right. \\
&\quad \left. + \sum_{j=1}^{i-1} (p_1 \cdots p_{j-1}) a_j (1 - b_j a_j^{-1} b_j^{-1}) z_j \otimes (p_1 \cdots p_{i-1}) y_i \right] - \\
&\quad - \sum_{i=1}^{n-1} \left[\sum_{j=1}^{i-1} \left((p_1 \cdots p_{j-1}) (1 - a_j b_j a_j^{-1}) y_j \otimes (p_1 \cdots p_{i-1}) a_i b_i a_i^{-1} y_i + \right. \right. \\
&\quad \left. \left. + (p_1 \cdots p_{j-1}) a_j (1 - b_j a_j^{-1} b_j^{-1}) z_j \otimes (p_1 \cdots p_{i-1}) a_i b_i a_i^{-1} y_i \right) + \right. \\
&\quad \left. + (p_1 \cdots p_{i-1}) (1 - a_i b_i a_i^{-1}) y_i \otimes (p_1 \cdots p_{i-1}) a_i b_i a_i^{-1} y_i + \right. \\
&\quad \left. + (p_1 \cdots p_{i-1}) a_i z_i \otimes (p_1 \cdots p_{i-1}) a_i b_i a_i^{-1} y_i \right] + \\
&\quad + \sum_{i=1}^n \left[\sum_{j=1}^{i-1} \left((p_1 \cdots p_{j-1}) (1 - a_j b_j a_j^{-1}) y_j \otimes (p_1 \cdots p_{i-1}) a_i z_i + \right. \right. \\
&\quad \left. \left. + (p_1 \cdots p_{j-1}) a_j (1 - b_j a_j^{-1} b_j^{-1}) z_j \otimes (p_1 \cdots p_{i-1}) a_i z_i \right) + \right. \\
&\quad \left. + (p_1 \cdots p_{i-1}) y_i \otimes (p_1 \cdots p_{i-1}) a_i z_i \right] -
\end{aligned}$$

$$\begin{aligned}
& - \sum_{i=1}^{n-1} \left[\sum_{j=1}^i (p_1 \cdots p_{j-1}) (1 - a_j b_j a_j^{-1}) y_j \otimes (p_1 \cdots p_i) z_i + \right. \\
& \quad \left. (p_1 \cdots p_{j-1}) a_j (1 - b_j a_j^{-1} b_j^{-1}) z_j \otimes (p_1 \cdots p_i) z_i \right] - \\
& - z_n \otimes b_n y_n.
\end{aligned}$$

Proof: There is a contracting homotopy s for the resolution P such that

$$\begin{aligned}
s_{-1}(1) &= x, \\
s_0(\tilde{p}x) &= \sum_{j=1}^n \left(\frac{\partial \tilde{p}}{\partial a_j} y_j + \frac{\partial \tilde{p}}{\partial b_j} z_j \right),
\end{aligned}$$

where \tilde{p} is a reduced word in G (we must, of course, be careful in our choice of \tilde{p} in order to keep ourselves from defining s_0 in two different ways for the same element of P_0). Once we have s_{-1} and s_0 , we can quickly compute $\Delta_0: P_0 \rightarrow P_0 \otimes P_0$ and $\Delta_1: P_1 \rightarrow (P \otimes P)_1$ using Proposition 2. We get

$$\begin{aligned}
\Delta_0(x) &= s_{-1}\varepsilon(x) \otimes s_{-1}\varepsilon(x) = x \otimes x, \\
\Delta_1(y_i) &= \tilde{s}_0 \Delta_0 d_1(y_i) = \tilde{s}_0(a_i x \otimes a_i x) - \tilde{s}_0(x \otimes x) \\
&= y_i \otimes a_i x + x \otimes y_i, \\
\Delta_1(z_i) &= \tilde{s}_0 \Delta_0 d_1(z_i) = \tilde{s}_0(b_i x \otimes b_i x) - \tilde{s}_0(x \otimes x) \\
&= z_i \otimes b_i x + x \otimes z_i.
\end{aligned}$$

Now, to calculate the maps $\Delta_{02}: P_2 \rightarrow P_0 \otimes P_2$ and $\Delta_{11}: P_2 \rightarrow P_1 \otimes P_1$, we observe the following: if $g, g' \in G$, then

$$\begin{aligned}
\tilde{s}_1(gx \otimes g'y_i) &= \underbrace{s_0(gx) \otimes g'y_i}_{\in P_1 \otimes P_1} + \underbrace{x \otimes s_1(g'y_i)}_{\in P_0 \otimes P_2}, \\
\tilde{s}_1(gx \otimes g'z_i) &= \underbrace{s_0(gx) \otimes g'z_i}_{\in P_1 \otimes P_1} + \underbrace{x \otimes s_1(g'z_i)}_{\in P_0 \otimes P_2}, \\
\tilde{s}_1(gy_i \otimes g'x) &= s_1(gy_i) \otimes g'x \in P_2 \otimes P_0, \\
\tilde{s}_1(gz_i \otimes g'x) &= s_1(gz_i) \otimes g'x \in P_2 \otimes P_0.
\end{aligned}$$

For the computation of Δ_{11} , we are interested in the terms that belong to $P_1 \otimes P_1$, and for Δ_{02} we want the terms belonging to $P_0 \otimes P_2$. So our knowledge of s_0 is sufficient for the calculation of Δ_{11} . Using Proposition 2, we get Δ_{11} as in the statement of the theorem.

Finally, to compute Δ_{02} , we need the map s_1 of our contracting homotopy s . The homomorphism s_1 must satisfy $s_1(d_2(w)) = w$, which is equivalent to

$$s_1 \left(\sum_{i=1}^n \frac{\partial p}{\partial a_i} y_i + \frac{\partial p}{\partial b_i} z_i \right) = w.$$

But we have

$$\begin{aligned}
\Delta_{02}(w) &= \pi_{02}\tilde{s}_1\Delta_1d_2(w) \\
&= \pi_{02}\tilde{s}_1\left(\frac{\partial p}{\partial a_i}(x \otimes y_i) + \frac{\partial p}{\partial b_i}(x \otimes z_i)\right) \\
&= x \otimes s_1\left(\sum_{i=1}^n \frac{\partial p}{\partial a_i}y_i + \frac{\partial p}{\partial b_i}z_i\right) = x \otimes w.
\end{aligned}$$

■

Corollary 1 *Let M be a trivial $\mathbb{Z}G$ -module. If $u, v \in \text{Hom}_{\mathbb{Z}G}(P_1, M)$, then the product $[u] \smile [v] \in H^2(G, M \otimes M)$ is represented by the map $(u \smile v) \in \text{Hom}_{\mathbb{Z}G}(P_2, M \otimes M)$ defined by*

$$(u \smile v)(w) = \sum_{i=1}^n (u(y_i) \otimes v(z_i) - u(z_i) \otimes v(y_i)).$$

Proof: We only need to notice that if M is a trivial $\mathbb{Z}G$ -module and $\varepsilon: \mathbb{Z}G \rightarrow \mathbb{Z}$ is the augmentation homomorphism, then for every $f \in \text{Hom}_{\mathbb{Z}G}(P_1, M)$ we have $f(k\alpha) = \varepsilon(k)f(\alpha)$, $\forall k \in \mathbb{Z}G, \forall \alpha \in P_1$. This, together with the formula we have for Δ_{11} , gives us the desired result.

■

Now we consider the case of the non-orientable surfaces: let N_n be the non-orientable surface with fundamental group given by the presentation

$$G = \pi_1(N_n) = \langle a_1, \dots, a_n \mid a_1^2 a_2^2 \cdots a_n^2 \rangle,$$

for $n \geq 2$. We also define $p = a_1^2 a_2^2 \cdots a_n^2$. The computations for the case of a non-orientable surface are similar to those we made in the orientable case, and so we simply state the free resolution and the diagonal approximation we get.

Proposition 4 (Free resolution for the non-orientable case) *A free resolution of \mathbb{Z} over $\mathbb{Z}G$ is given by*

$$0 \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \quad (3)$$

where

$$\begin{aligned}
P_0 &= \langle x \rangle \cong \mathbb{Z}G, \\
P_1 &= \langle y_1, \dots, y_n \rangle \cong \mathbb{Z}G^n, \\
P_2 &= \langle w \rangle \cong \mathbb{Z}G,
\end{aligned}$$

and the maps ε , d_1 and d_2 are defined by

$$\begin{aligned}
\varepsilon(x) &= 1, \\
d_1(y_i) &= (a_i - 1)x, \\
d_2(w) &= \sum_{i=1}^n \frac{\partial p}{\partial a_i} y_i,
\end{aligned}$$

where the partial derivatives are the Fox derivatives.

Theorem 2 (Diagonal approximation for the non-orientable case) *Let P be the free resolution of \mathbb{Z} over $\mathbb{Z}G$ given by Proposition 4. A diagonal approximation $\Delta: P \rightarrow (P \otimes P)$ is partially given by*

$$\begin{aligned}
\Delta_0: P_0 &\rightarrow (P \otimes P)_0 \\
\Delta_0(x) &= x \otimes x, \\
\Delta_1: P_1 &\rightarrow (P \otimes P)_1 \\
\Delta_1(y_i) &= y_i \otimes a_i x + x \otimes y_i, \\
\Delta_{02}: P_2 &\rightarrow (P_0 \otimes P_2) \\
\Delta_{02}(w) &= x \otimes w, \\
\Delta_{11}: P_2 &\rightarrow (P_1 \otimes P_1) \\
\Delta_{11}(w) &= \sum_{i=1}^n \left[\left(\sum_{j=1}^{i-1} (a_1^2 \cdots a_{j-1}^2)(1 + a_j)y_j \otimes (a_1^2 \cdots a_{i-1}^2)y_i \right) + \right. \\
&\quad \left. + \left(\sum_{j=1}^{i-1} (a_1^2 \cdots a_{j-1}^2)(1 + a_j)y_j \otimes (a_1^2 \cdots a_{i-1}^2)a_i y_i \right) + \right. \\
&\quad \left. + (a_1^2 \cdots a_{i-1}^2)y_i \otimes (a_1^2 \cdots a_{i-1}^2)a_i y_i \right].
\end{aligned}$$

3 Comments about applications

Now we make some comments about applications of the results of the previous section. The cohomology ring of a surface group with twisted coefficients has been used in many applications. As we've stated in the Introduction, see [5] for an example, where the local group is \mathbb{Q} , the rationals.

3.1 The Cohomology ring of a surface with arbitrary coefficients

In [8] the cohomology groups of a group which admits a presentation with one single relation was studied. This includes the surface groups for surfaces distinct from S^2 and $\mathbb{R}P^2$. More precisely in Corollary 11.2 they provide a formula for $H^n(G, K)$, $n \geq 2$. They prove the following result:

Corollary 2 ([8]) *If G is defined by a single relation R , where $R = Q^q$ for no $q > 1$, and if K is any left G -module, then $H^2(G, K) = K / \left(\frac{\partial R}{\partial x_1}, \dots, \frac{\partial R}{\partial x_m} \right) K$ and $H^n(G, K) = 0$ for all $n > 2$.*

Using the resolution of the previous section we can write a formula for H^1 . For, let G be a surface group with $G \neq \mathbb{Z}_2$, $G \neq \{1\}$ and let K be a left G -module. It is well known that $H^0(G, K) = K^G$, i.e. the set of elements of K which are fixed by the ring $\mathbb{Z}G$. If G is an orientable surface group, the group cohomology with coefficients K , from section 1, is given by the homology of the complex

$$0 \longrightarrow K \xrightarrow{d_1^*} K^{2n} \xrightarrow{d_2^*} K \longrightarrow 0, \quad (4)$$

where $d_1^*(k) = (a_1(k) - k, a_2(k) - k, \dots, a_n(k) - k, b_1(k) - k, \dots, b_n(k) - k)$ and $d_2^*(k_1, \dots, k_{2n}) = \sum_{i=1}^{i=n} \frac{\partial p}{\partial a_i}(k_i) + \frac{\partial p}{\partial b_i}(k_{i+n})$. We then get the following proposition:

Proposition 5 *For G an orientable surface group we have $H^1(G, K) = \text{Ker}(d_2^*)/d_1^*(K/K^G)$ where d_2^* is given as above.*

Remark: The calculation of the Fox derivative is straightforward. See, for example, [3].

Remark: If $G = \mathbb{Z} \oplus \mathbb{Z}$ and $K = \mathbb{Z} \oplus \mathbb{Z}$, let us consider the $\mathbb{Z}G$ -module structure given by $\theta(1, 0) = \theta(0, 1) \in GL_2(\mathbb{Z})$ equal to the matrix

$$\varphi = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since the matrix has no eigenvalue 1, follows that $H^0(G, K) = 0$. A direct calculation shows that d_2^* is surjective. So it follows that $H^2(G, K) = 0$. Also by a straightforward long calculation, we have $H^1(G, K) = 0$. This phenomena, i.e., the existence of a system of coefficients such that the cohomology is trivial, can not happen if the surface doesn't have Euler characteristic zero.

A similar result holds for the nonorientable surfaces and we leave the details to the reader.

3.2 Classification of Torus bundle over a surface

Let us consider the n -dimensional torus T^n , which is a Lie-group. There is a subfamily of the family of all torus bundles over a given surface whose elements are the principal torus bundles. It is well known that such bundles are classified by $[S, BT^n]$, (where BT^n is the classifying space of the torus), which in turn is $H^2(S, \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. In case we consider non-principal bundles, let us focus on those with a prescribed action $\theta : \pi_1(S) \rightarrow \text{Aut}(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) = GL_n(\mathbb{Z})$. Such bundles are classified and they are in one-to-one correspondence with $H^2(S, \mathbb{Z} \oplus \cdots \oplus \mathbb{Z})_\theta$, the second cohomology group with local coefficients given by θ . See Hilman [7], section 5, Lemma 5.1 and Robinson [9]. This group is well known and it is isomorphic to the quotient of $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ by the action, i.e., by the subgroups generated by the elements $\theta(g)(x) - x$ for all $g \in \pi_1(S)$ and $x \in (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})$. It is easy to find many actions θ such that the group $H^2(S, \mathbb{Z} \oplus \cdots \oplus \mathbb{Z})_\theta$ is trivial, i.e. there is only one bundle with that prescribed action.

3.3 Cohomology of surface bundles over a surface

The total space S of a surface bundle $S_1 \rightarrow S \rightarrow S_2$ over a surface S_2 is a 4-manifold. The calculation of the cohomology of those spaces can be approached using spectral sequences. The E_2 term of the spectral sequence is given by $H^p(S_2, H^q(S_1))$ where the cohomology is with local coefficients. The action of $\pi_1(S_2)$ on $H^q(S_1)$ is completely determined by the action on $H^1(S_1)$ and this is a data of the bundle. So one can make use of the section one to determine the ring structure of E_2 . We point out that this is a relevant step to find the cohomology of the space but in general not a sufficient one.

3.4 The Cohomology ring of a surface with arbitrary coefficients \mathbb{Z}

Here we compute the complete ring structure of the cohomology of a surface for any local system having as group the integers \mathbb{Z} . First we consider the case where G is the fundamental group of an orientable surface. If $\tilde{\mathbb{Z}}$ is a non-trivial G -module, then it is proved in [6] that the only action $\theta : G \rightarrow \text{Aut}(\mathbb{Z}) = \{1, -1\}$ we need to consider is the one given by $\theta(b_n)(1) = -1$, $\theta(a_i)(1) = 1$ for all $1 \leq i \leq n$ and $\theta(b_i)(1) = 1$ for all $1 \leq i \leq n-1$. We denote by $\tilde{\mathbb{Z}}$ the G -module \mathbb{Z} given by the action θ and denote by \mathbb{Z} the trivial G -module, and we also observe that $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, $\mathbb{Z} \otimes \tilde{\mathbb{Z}} \cong \tilde{\mathbb{Z}}$ and $\tilde{\mathbb{Z}} \otimes \tilde{\mathbb{Z}} \cong \mathbb{Z}$.

Theorem 3 Let $G = \langle a_1, b_1, a_2, b_2, \dots, a_n, b_n \mid [a_1, b_1][a_2, b_2] \cdots [a_n, b_n] \rangle$, for $n \geq 1$. The groups $H^*(G, \mathbb{Z})$ and $H^*(G, \tilde{\mathbb{Z}})$ are given by

$$\begin{aligned} H^0(G, \mathbb{Z}) &\cong \mathbb{Z}, & H^0(G, \tilde{\mathbb{Z}}) &= 0, \\ H^1(G, \mathbb{Z}) &\cong \mathbb{Z}^{2n}, & H^1(G, \tilde{\mathbb{Z}}) &\cong \mathbb{Z}^{2n-2} \oplus \mathbb{Z}_2, \\ H^2(G, \mathbb{Z}) &\cong \mathbb{Z}, & H^2(G, \tilde{\mathbb{Z}}) &\cong \mathbb{Z}_2. \end{aligned}$$

More precisely, in terms of the free resolution obtained in Proposition 3, we have the following sets of generators for the groups $H^*(G, \mathbb{Z})$ and $H^*(G, \tilde{\mathbb{Z}})$:

$$\begin{aligned} &\text{generator of } H^0(G, \mathbb{Z}) : \{[x^*]\}, \\ &\text{generators of } H^1(G, \mathbb{Z}) : \{[y_1^*], \dots, [y_n^*], [z_1^*], \dots, [z_n^*]\}, \\ &\text{generator of } H^2(G, \mathbb{Z}) : \{[w^*]\}, \\ &\text{generators of } H^1(G, \tilde{\mathbb{Z}}) : \{[y_1^*]_\theta, \dots, [y_{n-1}^*]_\theta, [z_1^*]_\theta, \dots, [z_n^*]_\theta\}, \quad (\text{remark: } [z_n^*]_\theta \text{ has order 2}) \\ &\text{generator of } H^2(G, \tilde{\mathbb{Z}}) : \{[w^*]_\theta\}, \end{aligned}$$

where $[\]$ and $[\]_\theta$ represent cohomology classes in $H^*(G, \mathbb{Z})$ and $H^*(G, \tilde{\mathbb{Z}})$, respectively. Finally, we have

$$\begin{aligned} [y_i^*]^2 &= 0 \quad \forall 1 \leq i \leq n, \\ [z_i^*]^2 &= 0 \quad \forall 1 \leq i \leq n, \\ [y_i^*] \smile [y_j^*] &= 0, \\ [z_i^*] \smile [z_j^*] &= 0, \\ [y_i^*] \smile [z_j^*] &= \begin{cases} [w^*], & \text{if } i = j \in \{1, \dots, n\}, \\ 0, & \text{if } i \neq j, \end{cases} \\ [y_i^*]_\theta^2 &= 0 \quad \text{for } 1 \leq i \leq n-1, \\ [z_i^*]_\theta^2 &= 0 \quad \text{for } 1 \leq i \leq n, \\ [y_i^*]_\theta \smile [z_i^*]_\theta &= [w^*] \quad \text{for } 1 \leq i \leq n-1, \\ [y_i^*]_\theta \smile [z_j^*]_\theta &= 0 \quad \text{if } i \neq j, \\ [y_i^*]_\theta \smile [y_j^*]_\theta &= 0 \quad \text{if } i \neq j, \\ [z_i^*]_\theta \smile [z_j^*]_\theta &= 0 \quad \text{if } i \neq j, \\ [y_i^*]_\theta \smile [y_j^*]_\theta &= 0 \quad \forall (i, j) \in \{1, \dots, n\} \times \{1, \dots, n-1\}, \\ [z_i^*]_\theta \smile [z_j^*]_\theta &= 0 \quad \forall i, j \in \{1, \dots, n\}, \\ [y_i^*]_\theta \smile [z_j^*]_\theta &= \begin{cases} [w^*]_\theta, & \text{if } i = j \in \{1, \dots, n-1\}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Proof: Follows by routine calculation using the diagonal approximation. ■

The non-orientable case is similar. Let $G = \langle a_1, \dots, a_n \mid a_1^2 a_2^2 \cdots a_n^2 \rangle$ for $n \geq 2$. In [6] it is proved that, if we want to consider all the possible structures of \mathbb{Z} as a G -module, then we only need to look at three different actions $\theta: G \rightarrow \text{Aut}(\mathbb{Z})$: the first one is the trivial action $\theta_0: G \rightarrow \text{Aut}(\mathbb{Z})$. The second one is $\theta_1: G \rightarrow \text{Aut}(\mathbb{Z})$, defined by $\theta_1(a_1)(1) = -1$ and $\theta_1(a_i)(1) = 1$ for $2 \leq i \leq n$. The third is $\theta_2: G \rightarrow \text{Aut}(\mathbb{Z})$, given by $\theta_2(a_1)(1) = \theta_2(a_2)(1) = -1$ and $\theta_2(a_i)(1) = 1$ for $3 \leq i \leq n$. We write \mathbb{Z}_{θ_i} for the G -module \mathbb{Z} determined by the action θ_i , for $0 \leq i \leq 2$. Observe that, for $0 \leq i \leq 2$, $\mathbb{Z}_{\theta_0} \otimes \mathbb{Z}_{\theta_i} \cong \mathbb{Z}_{\theta_i}$, $\mathbb{Z}_{\theta_i} \otimes \mathbb{Z}_{\theta_i} \cong \mathbb{Z}_{\theta_0}$, and $\mathbb{Z}_{\theta_1} \otimes \mathbb{Z}_{\theta_2} \cong \mathbb{Z}_{\theta_1}$.

Theorem 4 $G = \langle a_1, \dots, a_n \mid a_1^2 a_2^2 \cdots a_n^2 \rangle$, where $n \geq 2$. The cohomology groups $H^*(G, \mathbb{Z}_{\theta_i})$ are given by

$$\begin{aligned} H^0(G, \mathbb{Z}_{\theta_0}) &\cong \mathbb{Z}, & H^0(G, \mathbb{Z}_{\theta_1}) &= 0, & H^0(G, \mathbb{Z}_{\theta_2}) &= 0, \\ H^1(G, \mathbb{Z}_{\theta_0}) &\cong \mathbb{Z}^{n-1}, & H^1(G, \mathbb{Z}_{\theta_1}) &\cong \mathbb{Z}^{n-2} \oplus \mathbb{Z}_2, & H^1(G, \mathbb{Z}_{\theta_2}) &\cong \mathbb{Z}^{n-2} \oplus \mathbb{Z}_2, \\ H^2(G, \mathbb{Z}_{\theta_0}) &\cong \mathbb{Z}_2, & H^2(G, \mathbb{Z}_{\theta_1}) &\cong \mathbb{Z}_2, & H^2(G, \mathbb{Z}_{\theta_2}) &\cong \mathbb{Z}_2. \end{aligned}$$

More precisely, in terms of the free resolution obtained in Proposition 4, we can determine explicit generators for the groups $H^*(G, \mathbb{Z}_{\theta_i})$ as follows:

$$\begin{aligned} H^0(G, \mathbb{Z}_{\theta_0}) &= \langle [x^*]_{\theta_0} \rangle, \\ H^1(G, \mathbb{Z}_{\theta_0}) &= \bigoplus_{k=1}^{n-1} \langle [y_k^* - y_{k+1}^*]_{\theta_0} \rangle, \\ H^2(G, \mathbb{Z}_{\theta_0}) &= \langle [w^*]_{\theta_0} \rangle, \\ H^0(G, \mathbb{Z}_{\theta_1}) &= 0, \\ H^1(G, \mathbb{Z}_{\theta_1}) &= \langle [y_1^*]_{\theta_1} \rangle \oplus \left(\bigoplus_{k=2}^{n-1} \langle [y_k^* - y_{k+1}^*]_{\theta_1} \rangle \right), \\ H^2(G, \mathbb{Z}_{\theta_1}) &= \langle [w^*]_{\theta_1} \rangle, \\ H^0(G, \mathbb{Z}_{\theta_2}) &= 0, \\ H^1(G, \mathbb{Z}_{\theta_2}) &= \langle [y_1^*]_{\theta_2} \rangle \oplus \langle [y_1^* + y_2^*]_{\theta_2} \rangle \oplus \left(\bigoplus_{k=3}^{n-1} \langle [y_k^* - y_{k+1}^*]_{\theta_2} \rangle \right), \\ H^2(G, \mathbb{Z}_{\theta_2}) &= \langle [w^*]_{\theta_2} \rangle, \end{aligned}$$

where $[\]_{\theta_i}$ represents the cohomology class in $H^*(G, \mathbb{Z}_{\theta_i})$ and the elements $[y_1^*]_{\theta_1}$, $[y_1^* + y_2^*]_{\theta_2}$ and $[w^*]_{\theta_i}$ have order 2. Finally, the products

$$H^1(G, \mathbb{Z}_{\theta_i}) \otimes H^1(G, \mathbb{Z}_{\theta_j}) \xrightarrow{\sim} H^2(G, \mathbb{Z}_{\theta_i} \otimes \mathbb{Z}_{\theta_j})$$

are given by

$$[y_k^* - y_{k+1}^*]_{\theta_0}^2 = 0,$$

$$\begin{aligned}
[y_k^* - y_{k+1}^*]_{\theta_0} \smile [y_\ell^* - y_{\ell+1}^*]_{\theta_0} &= \begin{cases} [w^*]_{\theta_0}, & \text{if } \ell = k \pm 1, \\ 0, & \text{otherwise,} \end{cases} \\
[y_k^* - y_{k+1}^*]_{\theta_0} \smile [y_1^*]_{\theta_1} &= \begin{cases} [w^*]_{\theta_1}, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \\
[y_k^* - y_{k+1}^*]_{\theta_0} \smile [y_\ell^* - y_{\ell+1}^*]_{\theta_1} &= \begin{cases} [w^*]_{\theta_1}, & \text{if } \ell = k \pm 1, \\ 0, & \text{otherwise,} \end{cases} \\
[y_k^* - y_{k+1}^*]_{\theta_0} \smile [y_\ell^* - y_{\ell+1}^*]_{\theta_2} &= \begin{cases} [w^*]_{\theta_2}, & \text{if } \ell = k \pm 1, \\ 0, & \text{otherwise,} \end{cases} \\
[y_k^* - y_{k+1}^*]_{\theta_0} \smile [y_1^*]_{\theta_2} &= \begin{cases} [w^*]_{\theta_2}, & \text{if } k = 1, \\ 0, & \text{otherwise,} \end{cases} \\
[y_k^* - y_{k+1}^*]_{\theta_0} \smile [y_1^* + y_2^*]_{\theta_2} &= \begin{cases} [w^*]_{\theta_2}, & \text{if } k = 2, \\ 0, & \text{otherwise,} \end{cases} \\
[y_1^*]_{\theta_1}^2 &= [w^*]_{\theta_0}, \\
[y_k^* - y_{k+1}^*]_{\theta_1}^2 &= 0, \\
[y_1^*]_{\theta_1} \smile [y_k^* - y_{k+1}^*]_{\theta_1} &= 0, \\
[y_1^*]_{\theta_2}^2 &= [w^*]_{\theta_0}, \\
[y_1^* + y_2^*]_{\theta_2}^2 &= 0, \\
[y_1^*]_{\theta_2} \smile [y_1^* + y_2^*]_{\theta_2} &= [w^*]_{\theta_0}, \\
[y_1^*]_{\theta_2} \smile [y_k^* - y_{k+1}^*]_{\theta_2} &= 0, \\
[y_1^* + y_2^*]_{\theta_2} \smile [y_k^* - y_{k+1}^*]_{\theta_2} &= 0, \\
[y_k^* - y_{k+1}^*]_{\theta_0} \smile [y_\ell^* - y_{\ell+1}^*]_{\theta_1} &= \begin{cases} [w^*]_{\theta_1}, & \text{if } \ell = k \pm 1, \\ 0, & \text{otherwise,} \end{cases} \\
[y_1^*]_{\theta_1} \smile [y_1^*]_{\theta_2} &= [w^*]_{\theta_1}, \\
[y_1^*]_{\theta_1} \smile [y_1^* + y_2^*]_{\theta_2} &= 0, \\
[y_1^*]_{\theta_1} \smile [y_\ell^* - y_{\ell+1}^*]_{\theta_2} &= 0, \\
[y_k^* - y_{k+1}^*]_{\theta_1} \smile [y_1^*]_{\theta_2} &= 0, \\
[y_k^* - y_{k+1}^*]_{\theta_1} \smile [y_1^* + y_2^*]_{\theta_2} &= \begin{cases} [w^*]_{\theta_1}, & \text{if } k = 2, \\ 0, & \text{otherwise,} \end{cases} \\
[y_k^* - y_{k+1}^*]_{\theta_1} \smile [y_\ell^* + y_{\ell+1}^*]_{\theta_2} &= \begin{cases} [w^*]_{\theta_1}, & \text{if } \ell = k \pm 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof: Follows by routine calculation using the diagonal approximation. ■

Remark: The cohomology ring structure with arbitrary coefficients of the groups of the form $G \rtimes \mathbb{Z}_2$, where G is a surface group, seems an interesting problem. More precisely, those groups are natural groups to act freely and properly on even dimensional homotopy spheres. Hence certain cohomological properties of those groups are expected to show up. We hope to pursue this idea somewhere.

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